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Meadows—Algebraic Structures with Three or More Binary Operations*,†

ROBERT HAAS

Department of Pharmacology, School of Medicine, Case Western Reserve University, Cleveland, Ohio 44106

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Rings and fields, like the elementary arithmetic that they generalize, deal with two binary operations linked by a distributive law. The purpose of this paper is to explore algebraic structures with more than just two linked operations. These are produced here, in a very elementary and natural way, by simply adding further sequentially distributive operations to existing rings or fields; the resulting "overgrown" structures are termed meadows. A third operation can always be added to a finite field, and occasionally a fourth; Section 1 gives the complete construction and classification of finite meadows. Section 2 presents some three-operation meadows over the integers and the rational numbers, and Section 3 describes a meadow with an infinite set $*_n$, $n = \dots, -1, 0, 1, \dots$, of sequentially distributive operations, generated by ordinary addition and multiplication, over the real numbers. Section 4 demonstrates the essential uniqueness of the latter structure, and the concluding Section 5 has some remarks concerning the significance of the new binary operations, including possible relations to bifurcation theory and chemical kinetics.[1]

* In memory of Father Raymond W. Allen, S.J., 2/2/13–3/13/90.

† I have presented some of the results reported in this paper previously in talks at meetings of the Ohio section of the Mathematical Association of America, held at Bowling Green State University (1970) and John Carroll University (1991).

[1] "Polyrings" are another type of algebraic structure with multiple associative binary operations which have been studied previously [2]. Polyrings (which are always representable by systems of set transformations) are distinct from meadows because each of their operations is required to be (left) distributive over *all* lower operations.

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1. Finite Meadows

Suppose that R is a ring with operations $+$ and $\cdot$ and multiplicative identity 1. The aim is to add a third operation, $*$, which is associative and distributes over multiplication. Explicitly, for all elements a, b, and c, one requires associativity

$$(a * b) * c = a * (b * c) \tag{1}$$

and the distributive laws

$$(a \cdot b) * c = (a * c) \cdot (b * c) \tag{2}$$

$$c * (a \cdot b) = (c * a) \cdot (c * b). \tag{3}$$

If x is multiplicatively idempotent then, by (2) and (3), so are all $x * y$ and $y * x$. In particular, $0 * y$, $y * 0$, $1 * y$, and $y * 1$ must all be idempotent. This can be achieved by also requiring:

$$0 * y = y * 0 = 0 \qquad \text{(for all } y) \tag{4}$$

$$1 * y = y * 1 = 1 \qquad \text{(for all } y \neq 0). \tag{5}$$

Proposition 2 below shows that (4) and (5) actually follow from (1)–(3) in most cases of interest.

Rules (1)–(5) have surprisingly strong consequences for both R and $*$.

Proposition 1. *If a finite ring R admits an operation $*$ satisfying* (1)–(5), *then R is a field and $*$ is commutative.*

Proof. R has no multiplicative zero divisors: If $x, y \neq 0$ but $x \cdot y = 0$, then $0 = 0 * 1 = (x \cdot y) * 1 = (x * 1) \cdot (y * 1) = 1 \cdot 1 = 1$, a contradiction. Since R is finite, each nonzero element therefore has a multiplicative inverse; thus R is a finite division ring, or, by Wedderburn's theorem, a field. Its multiplicative group of nonzero elements is therefore cyclic. Let t be a generator; then any nonzero x and y may be written $x = t^n$, $y = t^m$, and, by induction from (2) and (3), $x * y = t^n * t^m = (t * t)^{nm} = y * x$.

Conversely, rules (4)–(5) almost always follow from (1)–(3) in a field.

Proposition 2. *An operation $*$ in a field which satisfies* (1)–(3) *must also satisfy* (4) *and* (5), *except in five exceptional trivial cases: when $x * y$ is* (i) *always* 1, (ii) 0 *when $x = y = 0$ and* 1 *otherwise*, (iii) 0 *when $x = 0$ and* 1 *otherwise*, (iv) 0 *when $y = 0$ and* 1 *otherwise, or* (v) *always* 0.

i	
	1

ii	0	
0	0	
		1

iii	
0	0 0 0 0 0
	1

iv	0	
	0	
	0	
	0	1
	0	
	0	

v	
	0

Furthermore, except in Case v, $x \neq 0$ *and* $y \neq 0$ *implies* $x * y \neq 0$.

Proof. It is easy to check that, in each of the cases i–v, $*$ is associative and distributes over multiplication.

The terms $x * 0$, $0 * y$, $x * 1$, and $1 * y$ are idempotent, hence 0 or 1. If $0 * 0 = 1$, then $x * y = 1$ for all x and y (Case i) because $1 = (0 \cdot x) * (0 \cdot y) = (0 * 0) \cdot (x * 0) \cdot (0 * y) \cdot (x * y)$.

Assume, therefore, that $0 * 0 = 0$. If $a * 0 = 1$ for some a, then $x * 0 = 1$ for any $x \neq 0$, because $1 = a * 0 = (a * 0) \cdot (x^{-1} * 0) \cdot (x * 0)$. In consequence, for any y, $x * y = (x * y) \cdot (x * 0) = x * (y \cdot 0) = 1$. Similarly, $0 * b = 1$ for some b yields $x * y = 1$ for all x and y ($y \neq 0$). The occurrence of one or both of these possibilities yields Cases ii, iii, and iv; otherwise, $a * 0 = 0 * b = 0$ (Eq. 4).

If $a * b = 0$ for some $a \neq 0$, $b \neq 0$, then $x * y = 0$ for all x and y (Case v) because $a * b$ is a factor in $x * y = (x \cdot a \cdot a^{-1}) * (y \cdot b \cdot b^{-1})$. This proves Eq. (5) and the last statement of the proposition.

The structures described in Proposition 2 are termed trivial meadows. Nontrivial ones, satisfying (1)–(5), are constructed as follows.

THEOREM 1. *All nontrivial meadows over a finite field R with multiplicative generator t may be constructed as follows: Define $t * t = k$, where k is a fixed nonzero element of R, and extend the definition of $*$ by $t^n * t^m = (t * t)^{nm}$, for any positive integers n and m.*

Proof. If R forms a meadow, $t * t$ must be some $k \in R$, and $t^n * t^m = (t * t)^{nm}$ by distributivity. Conversely, if $*$ is defined as prescribed, then routine calculations show it is well-defined, associative, and distributive as required.

Since any automorphism of a finite field will induce a meadow automorphism (and conversely), the distinct meadows correspond to orbits $\{k, k^p, k^{p^2}, \ldots\}$ under the Frobenius automorphism $x \to x^p$, where p is the characteristic of the field.

Tables I and II present examples of the construction, the meadows of order 5 and order 9.

In rare cases a finite meadow can have four operations.

TABLE I

Meadows of Order 5

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

k = 0

*	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	0	0	0
3	0	0	0	0	0
4	0	0	0	0	0

k = 1

*	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1
2	0	1	1	1	1
3	0	1	1	1	1
4	0	1	1	1	1

k = 2

*	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1
2	0	1	2	3	4
3	0	1	3	2	4
4	0	1	4	4	1

k = 3

*	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1
2	0	1	3	2	4
3	0	1	2	3	4
4	0	1	4	4	1

k = 4

*	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1
2	0	1	4	4	1
3	0	1	4	4	1
4	0	1	1	1	1

Note. The first two tables are addition and multiplication in Z_5; the rest give $*$ as determined by the choice of the term $k = 2 * 2$.

THEOREM 2. *Suppose that* $+, \cdot, *$, *and* $**$ *are binary operations on a finite set* R *of order* $n > 3$ *and that* $(R, +, \cdot, *)$ *and* $(R', \cdot, *, **)$ *form nontrivial meadows* ($R' = R \sim \{0\}$). *This can happen if and only if* n *is* 1 *greater than a Mersenne prime, that is,* $n = 2^q$, *where both* q *and* $n - 1$ *are prime.*

Proof. Let $(R, +, \cdot, *)$ and $(R', \cdot, *, **)$ be nontrivial meadows; then $(R, +, \cdot)$ is a finite field of prime power order $n = p^r$, and $(R', \cdot)$ is a cyclic group, say with generator t. $n - 1$ is prime, because if $n - 1 = gh$ then $t^g * t^h = (t * t)^{gh} = 1$, i.e., the field $(R', \cdot, *)$ would have zero-divisors. Note that $r > 1$, else n and $n - 1$ would both be primes, forcing $n = 3$. But Mersenne primes $2^q - 1$ (q prime) are the only primes of the form $a^r - 1$ with $r > 1$, because $a^r - 1$ is divisible by $a - 1$ if $a > 2$, and $2^{cd} - 1$ is divisible by $2^c - 1$.

TABLE II

Meadows of Order 9

+	0	1	2	x	x + 1	x + 2	2x	2x + 1	2x + 2
0	0	1	2	x	x + 1	x + 2	2x	2x + 1	2x + 2
1	1	2	0	x + 1	x + 2	x	2x + 1	2x + 2	2x
2	2	0	1	x + 2	x	x + 1	2x + 2	2x	2x + 1
x	x	x + 1	x + 2	2x	2x + 1	2x + 2	0	1	2
x + 1	x + 1	x + 2	x	2x + 1	2x + 2	2x	1	2	0
x + 2	x + 2	x	x + 1	2x + 2	2x	2x + 1	2	0	1
2x	2x	2x + 1	2x + 2	0	1	2	x	x + 1	x + 2
2x + 1	2x + 1	2x + 2	2x	1	2	0	x + 1	x + 2	x
2x + 2	2x + 2	2x	2x + 1	2	0	1	x + 2	x	x + 1

·	0	1	2	x	x + 1	x + 2	2x	2x + 1	2x + 2
0	0	0	0	0	0	0	0	0	0
1	0	1	2	x	x + 1	x + 2	2x	2x + 1	2x + 2
2	0	2	1	2x	2x + 2	2x + 1	x	x + 2	x + 1
x	0	x	2x	2	x + 2	2x + 2	1	x + 1	2x + 1
x + 1	0	x + 1	2x + 2	x + 2	2x	1	2x + 1	2	x
x + 2	0	x + 2	2x + 1	2x + 2	1	x	x + 1	2x	2
2x	0	2x	x	1	2x + 1	x + 1	2	2x + 2	x + 2
2x + 1	0	2x + 1	x + 2	x + 1	2	2x	2x + 2	x	1
2x + 2	0	2x + 2	x + 1	2x + 1	x	2	x + 2	1	2x

$k = 0$

*	0	1	2	x	x + 1	x + 2	2x	2x + 1	2x + 2
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0
x	0	0	0	0	0	0	0	0	0
x + 1	0	0	0	0	0	0	0	0	0
x + 2	0	0	0	0	0	0	0	0	0
2x	0	0	0	0	0	0	0	0	0
2x + 1	0	0	0	0	0	0	0	0	0
2x + 2	0	0	0	0	0	0	0	0	0

$k = 1$

*	0	1	2	x	x + 1	x + 2	2x	2x + 1	2x + 2
0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1	1
2	0	1	1	1	1	1	1	1	1
x	0	1	1	1	1	1	1	1	1
x + 1	0	1	1	1	1	1	1	1	1
x + 2	0	1	1	1	1	1	1	1	1
2x	0	1	1	1	1	1	1	1	1
2x + 1	0	1	1	1	1	1	1	1	1
2x + 2	0	1	1	1	1	1	1	1	1

TABLE II—*Continued*

k = 2

*	0	1	2	x	x + 1	x + 2	2x	2x + 1	2x + 2
0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1	1
2	0	1	1	1	1	1	1	1	1
x	0	1	1	1	1	1	1	1	1
x + 1	0	1	1	1	2	2	1	2	2
x + 2	0	1	1	1	2	2	1	2	2
2x	0	1	1	1	1	1	1	1	1
2x + 1	0	1	1	1	2	2	1	2	2
2x + 2	0	1	1	1	2	2	1	2	2

k = x

*	0	1	2	x	x + 1	x + 2	2x	2x + 1	2x + 2
0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1	1
2	0	1	1	1	1	1	1	1	1
x	0	1	1	1	2	2	1	2	2
x + 1	0	1	1	2	x	2x	2	2x	x
x + 2	0	1	1	2	2x	x	2	x	2x
2x	0	1	1	1	2	2	1	2	2
2x + 1	0	1	1	2	2x	x	2	x	2x
2x + 2	0	1	1	2	x	2x	2	2x	x

k = x + 1

*	0	1	2	x	x + 1	x + 2	2x	2x + 1	2x + 2
0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1	1
2	0	1	1	1	2	2	1	2	2
x	0	1	1	2	x	2x	2	2x	x
x + 1	0	1	2	x	x + 1	x + 2	2x	2x + 1	2x + 2
x + 2	0	1	2	2x	x + 2	x + 1	x	2x + 2	2x + 1
2x	0	1	1	2	2x	x	2	x	2x
2x + 1	0	1	2	2x	2x + 1	2x + 2	x	x + 1	x + 2
2x + 2	0	1	2	x	2x + 2	2x + 1	2x	x + 2	x + 1

k = x + 2

*	0	1	2	x	x + 1	x + 2	2x	2x + 1	2x + 2
0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1	1
2	0	1	1	1	2	2	1	2	2
x	0	1	1	2	2x	x	2	x	2x
x + 1	0	1	2	2x	x + 2	x + 1	x	2x + 2	2x + 1
x + 2	0	1	2	x	x + 1	x + 2	2x	2x + 1	2x + 2
2x	0	1	1	2	x	2x	2	2x	x
2x + 1	0	1	2	x	2x + 2	2x + 1	2x	x + 2	x + 1
2x + 2	0	1	2	2x	2x + 1	2x + 2	x	x + 1	x + 2

Note. The field of order 9 is the algebraic extension field over Z_3 generated by a root of the irreducible polynomial $x^2 + 1$. The element $y = x + 1$ is a multiplicative generator, and $*$ is determined by the choice of $k = y * y$, which may be 0 (trivial), 1, 2, x (equivalently, $2x$), $x + 1$ (equivalently, $2x + 1$), or $x + 2$ (equivalently, $2x + 2$).

Conversely, let $n = 2^q$ be 1 greater than a Mersenne prime. Let $(R, +, \cdot)$ be a finite field of order n, denote $R' = R \sim \{0\}$ and $R'' = R' \sim \{1\}$, and let t generate $(R', \cdot)$, a cyclic group of prime order $n - 1$. By Theorem 1, $(R, +, \cdot)$ can be extended to a nontrivial meadow $(R, +, \cdot, *)$ in which $t * t = t$. If $x = t^a$ and $y = t^b$ are $\neq 1$, then $x * y = (t * t)^{ab} = t^{ab} \neq 1$ since t has prime order. $(R', \cdot, *)$ is therefore a finite ring without zero divisors, hence by Wedderburn's theorem a field. (R'' is not known *a priori* to have an identity under $*$, but $(R'', *)$ is a group because, being closed under $*$, it is a finite semigroup in which the cancellation laws hold; thus $(R', \cdot, *)$ is a finite division ring as required.) By Theorem 1, $(R', \cdot, *)$ may therefore be extended to a nontrivial meadow $(R', \cdot, *, **)$.

Table III shows one of the twelve nontrivial meadows with four operations of order 8. There are evidently no similar structures with five operations, since this would require that two consecutive numbers be Mersenne primes.

2. Infinite Meadows: The Integers and the Rational Numbers

The finite meadows over a field with generator t were constructed in Theorem 1.1 by using "multiplicative linearity," $t^n * t^m = (t * t)^{nm}$, to extend a definition $t * t = k$ over all pairs. For integer meadows one begins similarly by assigning an integer $k_{pq} = p * q$ for each pair of primes and extending the definition by multiplicative linearity. Thus, if nonzero integers a and b have unique prime decompositions $a = (-1)^{c_0} p_1^{c_1} p_2^{c_2} \ldots p_n^{c_n}$ and $b = (-1)^{d_0} q_1^{d_1} q_2^{d_2} \ldots q_m^{d_m}$, where c_0 and d_0 are 0 or 1, let[2]

$$a * b = \prod_{i,j} (k_{p_i q_j})^{c_i d_j}. \tag{1}$$

Evidently $*$ distributes over multiplication if and only if relation (1) is satisfied.

But when will $*$ also be associative? Clearly it is sufficient to test $p * (q * r) = (p * q) * r$ when p, q, and r are prime, -1, or 0. Suppose now for simplicity that the k_{pq}'s are all positive. Then k_{pq} has a unique prime factorization,

$$k_{pq} = \prod_i p_i^{c_i(p,q)}, \tag{2}$$

[2] Note that -1 has been included among the primes; in a nontrivial meadow, k_{pq} must always be ± 1 when p or q is -1 because, for instance, $[p * (-1)]^2 = p * [(-1)^2] = p * 1 = 1$.

TABLE III

A Meadow with Four Operations of Order 8

+	A	B	C	D	E	F	G	H
A	A	B	C	D	E	F	G	H
B	B	A	D	C	F	E	H	G
C	C	D	A	B	G	H	E	F
D	D	C	B	A	H	G	F	E
E	E	F	G	H	A	B	C	D
F	F	E	H	G	B	A	D	C
G	G	H	E	F	C	D	A	B
H	H	G	F	E	D	C	B	A

·	A	B	C	D	E	F	G	H
A	A	A	A	A	A	A	A	A
B	A	B	C	D	E	F	G	H
C	A	C	E	G	F	H	B	D
D	A	D	G	F	B	C	H	E
E	A	E	F	B	H	D	C	G
F	A	F	H	C	D	G	E	B
G	A	G	B	H	C	E	D	F
H	A	H	D	E	G	B	F	C

*	A	B	C	D	E	F	G	H
A	A	A	A	A	A	A	A	A
B	A	B	B	B	B	B	B	B
C	A	B	C	D	E	F	G	H
D	A	B	D	H	F	C	E	G
E	A	B	E	F	H	G	D	C
F	A	B	F	C	G	E	H	D
G	A	B	G	E	D	H	C	F
H	A	B	H	G	C	D	F	E

**	A	B	C	D	E	F	G	H
A	A	A	A	A	A	A	A	A
B	A	B	B	B	B	B	B	B
C	A	B	C	C	C	C	C	C
D	A	B	C	D	E	F	G	H
E	A	B	C	E	E	H	C	H
F	A	B	C	F	H	D	G	E
G	A	B	C	G	C	G	G	C
H	A	B	C	H	H	E	C	E

Note. The elements 0, 1, x, $x + 1$, x^2, $x^2 + 1$, $x^2 + x$, and $x^2 + x + 1$ of the extension field of Z_2 generated by a root of the irreducible polynomial $x^3 + x^2 + 1$ are represented here by the letters A, B, C, D, E, F, G, and H, respectively. The operation * is determined by the choice C * C = C, and ** by the choice D ** D = D.

and $p * (q * r) = (p * q) * r$ is equivalent to

$$\sum_i c_j(p, p_i) \cdot c_i(q, r) = \sum_i c_j(p_i, r) \cdot c_i(p, q) \qquad \text{(for each } j\text{).} \tag{3}$$

This set of equations appears difficult to solve in full generality, and three special solutions are given here as examples instead.

I. Let the matrix $\{k_{pq}\}$ be 1 except on its main diagonal,

$$k_{pq} = \begin{cases} p^{\alpha_p} & \text{if } p = q \\ 1 & \text{if } p \neq q, \end{cases} \tag{4}$$

where the α_p's are arbitrary nonnegative integers. $a * b$ is thus a number with prime factors dividing both a and b; the exponent of the prime p is the product of α_p and the corresponding exponents of p in a and b (much as for the finite meadows of Section 1).

II. Allow nontrivial off-diagonal entries in $\{k_{pq}\}$, but require all values to be powers of one fixed prime s:

$$k_{pq} = s^{\sqrt{\alpha_p \alpha_q}}, \tag{5}$$

where the α_p's are nonnegative integers such that, for any p and q, $\alpha_p \alpha_q$ is a square. Solution (5) can be shown to be the only one of the form $k_{p,q} = s^{\alpha_{p,q}}$ with all $\alpha_{p,q}$ positive.

III. (A simple noncommutative case).

$$k_{pq} = \begin{cases} 2 & \text{if } p = 5 \text{ and } q = 3 \\ 1 & \text{otherwise.} \end{cases} \tag{6}$$

Any solution of (3), for example (4), (5), or (6), can then be extended by (1) to give a meadow over the integers.

Note, incidentally, that the same approach can also be applied in other unique factorization domains. For instance, the polynomials in x over the integers form a meadow if one defines, for all irreducible polynomials $p(x)$ and $q(x)$ with positive leading coefficients,

$$p(x) * q(x) = \begin{cases} [p(x)]^{\alpha_p} & \text{if } p(x) = q(x) & \text{(nonconstant)} \\ 1 & \text{if } p(x) \neq q(x) & \text{(both nonconstant)} \\ 1 & \text{if } p(x) \text{ or } q(x) \text{ is constant} & \text{(both nonzero),} \end{cases}$$

where the α_p's are arbitrary nonnegative integers.

Integer meadows now extend directly over the rational numbers.

PROPOSITION 1. *Any nontrivial meadow over an integral domain extends uniquely to a meadow over the quotient field.*

Proof. $*$ defined on an integral domain extends to a well-defined associative, distributive operation on the quotient field by the formula

$$\frac{a}{b} * \frac{c}{d} = \frac{(a * c) \cdot (b * d)}{(a * d) \cdot (b * c)}. \tag{7}$$

For example, when formula (7) is applied to the integer meadow defined by (4) with α_t always 1, then for distinct primes p, q, r, and s,

$$\frac{p}{q} * \frac{p}{q} = pq, \ \frac{p}{q} * \frac{p}{r} = p, \ \frac{p}{q} * \frac{r}{s} = 1, \ \frac{p}{q} * \frac{r}{p} = \frac{1}{p}, \text{ and } \frac{p}{q} * \frac{q}{p} = \frac{1}{pq}.$$

3. Infinite Meadows: The Real Numbers

This section turns to the problem of constructing a meadow over the real numbers. The development here is self-contained; however, as described in Section 4, it also follows from the general theory of distributive pairs of binary operations on the real numbers (cf. [1]).

For a meadow with three operations, one seeks a third operation $*$ distributing over ordinary multiplication. For positive real numbers, multiplication itself may be *defined* in terms of addition:

$$a \cdot b = \exp(\ln a + \ln b) \qquad (a, b > 0). \tag{1}$$

Guided by this analogy, define $*$ by

$$a * b = \exp(\ln a \cdot \ln b) \qquad (a, b > 0). \tag{2}$$

(Theorem 4.1 below shows that the definition of $*$ is no mere analogy, but is actually determined up to a constant from distributivity and continuity.)[3]

Evidently $*$ is associative, commutative, and distributes over multiplication; its identity element is e, and the inverse to $x \neq 1$ is $\exp(1/\ln x)$. $z = x * y$ is continuous as a real-valued function of its two arguments; it may be realized graphically by replacing the three axes of a plot of $z = x \cdot y$ with ln scales.

The following proposition suggests how to extend the definition of $*$ to negative numbers.

Proposition 1. *If $*$ is a binary operation on the real numbers which distributes over multiplication, and $x > 0$, $y > 0$, then*

(i) $x * y = x * (-y) = (-x) * y$,

(ii) $(-x) * (-y)$ *is either always* $x * y$, *or always* $-(x * y)$.

Proof. (i) Both terms $x * y$ and $x * (-y)$ are positive because, for any number z, $x * z = (\sqrt{x} * z)^2 \geq 0$; they are equal because $[x * (-y)]^2 = x * [(-y)^2] = x * y^2 = (x * y)^2$.

(ii) From part (i) and distributivity, $(-x) * (-y) = [(-1) * (-1)] \cdot [(-1) * y] \cdot [x * (-y)] = [(-1) * (-1)] \cdot (1 * y) \cdot (x * y) = [(-1) * (-1)] \cdot (x * y)$. Conclusion (ii) follows vacuously if $*$ is always 0. Otherwise

[3] The main step in constructing both finite (Section 1) and integer meadows (Section 2) was to express an arbitrary $a * b$ in the form $g^n * g^m$ or $(p_0^{c_0} \dots p_n^{c_n}) * (q_0^{d_0} \dots q_m^{d_m})$ and operate on the exponents. In effect, one was already then working with the "logarithms" of a and b.

(Proposition 1.2), $(-1) * 1 = 1$ and $[(-1) * (-1)] = \pm 1$ because $[(-1) * (-1)]^2 = (-1) * [(-1)^2] = (-1) * 1 = 1$, completing the proof.

According to the two possibilities of Proposition 1(ii), one therefore defines

$$a * b = \begin{cases} \exp(\ln |a| \cdot \ln |b|) & (a, b \neq 0) \\ 0 & (a \text{ or } b = 0) \end{cases} \tag{3}$$

and

$$a *^- b = \begin{cases} \sigma(a, b) \exp(\ln |a| \cdot \ln |b|) & (a, b \neq 0) \\ 0 & (a \text{ or } b = 0), \end{cases} \tag{4}$$

where $\sigma(a, b) = -1$ when $a < 0$ and $b < 0$, $\sigma(a, b) = 1$ otherwise. Both $*$ and $*^-$ are associative, commutative, and distribute over multiplication; $*$ is illustrated graphically in Fig. 1e below.

Finite meadows can have at most four operations. Far more is possible with the real numbers: the method that gave $*$ extends to produce a whole doubly infinite family of associative, commutative, binary operations $*_i$, each distributing over the preceding one. The ascending terms are simply defined recursively by $*_1 = \cdot$, $*_2 = *$, and

$$a *_{i+1} b = \begin{cases} \exp(\ln |a| *_i \ln |b|) & (a, b \neq 0) \\ 0 & (a \text{ or } b = 0). \end{cases} \tag{5}$$

To express this in closed form, let $\operatorname{Ln} x = \ln |x|$, let $f^{(n)}$ denote the nth iteration of f $(n \geq 1)$ and $f^{(0)}$, the identity function, and let $e_i = \exp^{(i)}(0)$ for $i \geq 0$. Also, let

$$D_n = \begin{cases} (e_{n-1}, \infty) & \text{if } n \geq 1 \\ (-\infty, \infty) & \text{if } n \leq 0. \end{cases}$$

Then, for $n = 0, 1, 2, \ldots$,

$$a *_{n+1} b = \begin{cases} \exp^{(n)}[\operatorname{Ln}^{(n)} a \cdot \operatorname{Ln}^{(n)} b] & \text{if } \operatorname{Ln}^{(n)} a \text{ and } \operatorname{Ln}^{(n)} b \text{ exist} \\ e_i & \text{if, for } i < n,\ \operatorname{Ln}^{(i)} a \text{ exists and } \operatorname{Ln}^{(i)} b = 0, \text{ or } \operatorname{Ln}^{(i)} b \text{ exists and } \operatorname{Ln}^{(i)} a = 0. \end{cases} \tag{6}$$

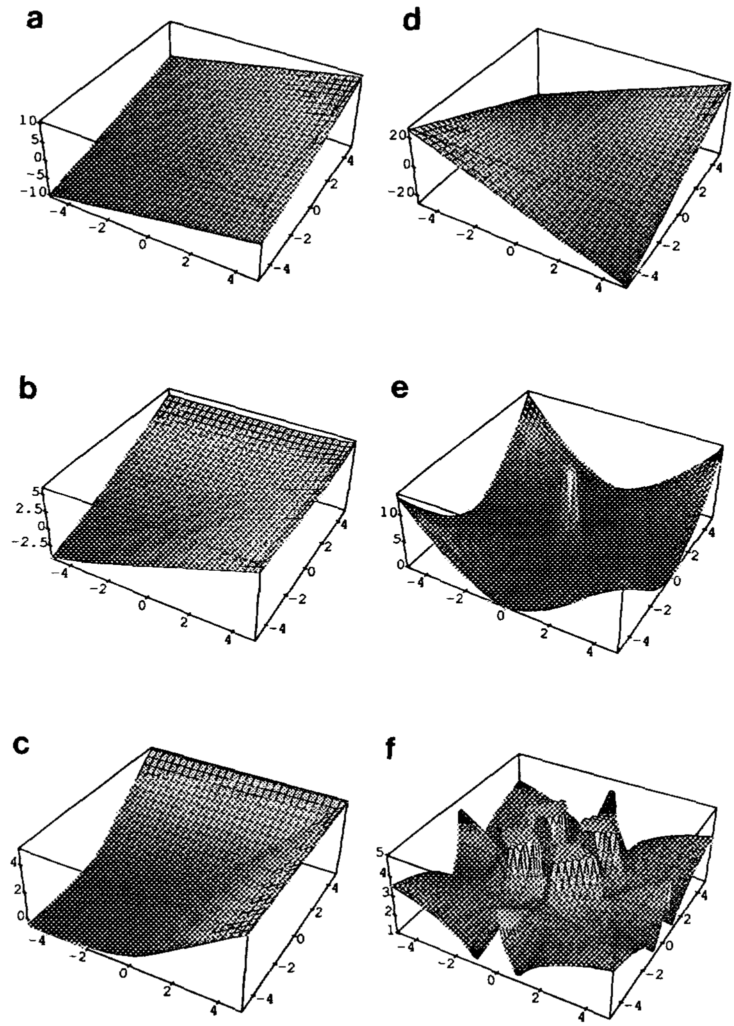

FIG. 1. Meadows over the real numbers. The function $z = x *_n y$ has been graphed here for $n = 0$ (panel a), -1 (panel b), -2 (panel c), 1 (panel d), 2 (panel e), and 3 (panel f), using the computer program Mathematica (Wolfram Research, Inc.) [9]. Note that the z axis scales differ among panels. The plots are truncated at offscale singularities in panels e and f; also, the values $x *_3 0 = 0 *_3 x = 0$ below the range in panel f are not shown. The "mesh" (number of points evaluated) was 30 per side in panels a–d and 50 (giving little change in overall shape from 30) in e and f. The plot in panel f should actually also diverge to $+\infty$ along the lines $x = 0$ ($|y| > e$) and $y = 0$ ($|x| > e$), but the program did not graph it accurately because $x *_3 y$ rises so abruptly (e.g., $5 *_3 0.01$, $4 *_3 0.00001$, and $3 *_3 10^{-3000}$ are all < 10): the partition the graph forms between quadrants can thus be 3000 orders of magnitude thinner than the unit distance of the plot.

The descending terms $*_n$, for $n = 0, -1, -2, \ldots$, are defined:

$$a *_n b = \ln^{(-n)}[\exp^{(-n)}a + \exp^{(-n)}b] \tag{7}$$

or, equivalently,

$$a *_{n+1} b = \ln^{(-n)}[\exp^{(-n)}a \cdot \exp^{(-n)}b]. \tag{8}$$

Graphs of $x *_n y$ for some values of n are shown in Fig. 1.

As for $x * y$ above, $x *_{n+1} y$ may also be realized graphically simply by changing the axes of a plot of $z = x \cdot y$ to $\ln^{(n)}$ scales ($n \geq 0$; $x, y \in D_n$), or $\exp^{(-n)}$ scales ($n \leq 0$). The function $x *_{n+1} y$ therefore increases very rapidly with x and y for large positive n, but is greatly flattened towards $\max(x, y)$ when n is negative.

The following theorem summarizes the properties of the $*_n$.

THEOREM 1. *Define $*_n$ over the real numbers by* (5) *or* (6) *and* (7) *or* (8). *Then $*_0$ is ordinary addition, $*_1$ is ordinary multiplication, and each $*_n$ is associative, commutative, and distributes over $*_{n-1}$. When $n \geq 0$, $*_n$ in the domain $D_{n-1} \times D_{n-1}$ has e_n as an identity element, and the inverse to x is $-x$ ($n = 0$), or* $\exp^{(n-1)}[1/\ln^{(n-1)}x]$ ($n \geq 1$, $x \neq e_{n-1}$). *For all n, $x *_n y$ is a continuous function of x and y, except for possible singularities when $n \geq 2$ and* $\mathrm{Ln}^{(n-1)}x$ *or* $\mathrm{Ln}^{(n-1)}y$ *is not defined. In particular, $*_n$: $D_{n-1} \times D_{n-1} \rightarrow D_{n-1}$ is continuous.*

Are there well-behaved binary operations on the real numbers beyond ordinary addition and multiplication? This theorem shows the answer is: Yes, infinitely many!

4. UNIQUENESS OF THE $*_n$'S

This section demonstrates that each $*_{n+1}$ is determined up to a constant from $*_n$, so that the set of operations $*_n$ is essentially unique. Two proofs are given: one based on the quite general description of distributive operations available from the theory of functional equations, and an alternative, self-contained development that has the independent interest of generalizing the power function.

Any continuous group operation on an interval of real numbers must have the form

$$G(x, y) = h^{-1}[h(x) + h(y)],$$

where h is continuous and strictly monotonic [1, pp. 57, 254]. It is then easy to prove that a continuous commutative operation F will distribute over G if and only if it has the form

$$F(x, y) = h^{-1}[k \cdot h(x)h(y)],$$

where k is some fixed constant (cf. [1, p. 318]). For example, $G = *_n$ $(n > 0)$ when $h = \ln^{(n)}$ in the domain (e_{n-1}, ∞), and an operation like $*_{n+1}$ distributing over $*_n$ is then determined up to the constant k.

For an alternative proof leading to the same conclusion (Theorem 1 below), set up notation $\langle x\rangle^k_{*_n}$ to represent the k-fold iteration of $*_n$, in the same way that x^k represents iterated multiplication. Just as x^k is defined in two different ways, as a k-fold product of x's when k is a positive integer, and, for $x > 0$, as $e^{k \cdot \ln x}$ when k is an arbitrary real number, define

$$\langle x\rangle^k_{*_n} = \underbrace{x *_n x *_n \dots *_n x}_{k \text{ times}} \tag{1}$$

when k is a positive integer, and

$$\langle x\rangle^k_{*_n} = \begin{cases} \exp^{(n)}[k \cdot \ln^{(n)} x] & \text{if } n \geq 0 \\ \ln^{(-n)}[k \cdot \exp^{(-n)} x] & \text{if } n \leq 0 \end{cases} \tag{2}$$

for $x \in D_n$, when $k \in E_n(x)$ defined as

$$E_n(x) = \begin{cases} (-\infty, \infty) & \text{if } n \geq 0 \\ (e_{-n-1}/\exp^{(-n)} x, \infty) & \text{if } n < 0. \end{cases}$$

(Note that, as with the pair of formulas (3.6) and (3.8), the two parts of (2) yield a single formula if one agrees to write $\ln^{(-n)} = \exp^{(n)}$.) Routine calculations then establish that (1) and (2) coincide whenever k is a positive integer and $x \in D_n$.

$\langle x\rangle^k_{*_n}$ generalizes ordinary exponents in several different ways. For example, $\langle x\rangle^k_{*_0} = kx$ and $\langle x\rangle^k_{*_1} = x^k$; $\langle x\rangle^0_{*_n} = e_n$ $(n \geq 0)$ and $\langle x\rangle^1_{*_n} = x$; $\langle e_n\rangle^k_{*_n} = e_n$ $(n \geq 0)$, and $\langle e_{n+1}\rangle^k_{*_n}$ is $\exp^{(n)} k$ when $n \geq 0$, and $\ln k$ when $n = -1$. The familiar sum and product rules of exponents also have their analogues.

PROPOSITION 1.

(a) $\langle x\rangle^k_{*_n} *_n \langle x\rangle^j_{*_n} = \langle x\rangle^{k+j}_{*_n}$

(b) $\langle\langle x\rangle^k_{*_n}\rangle^j_{*_n} = \langle x\rangle^{kj}_{*_n}$

(c) $\langle x\rangle^k_{*_n} *_n \langle y\rangle^k_{*_n} = \langle x *_n y\rangle^k_{*_n}$
(d) $\langle x\rangle^j_{*_n} *_{n+1} \langle y\rangle^k_{*_n} = \langle x *_{n+1} y\rangle^{j\cdot k}_{*_n}$
(e) $\langle\langle x\rangle^j_{*_n}\rangle^k_{*_{n+1}} = \langle\langle x\rangle^k_{*_{n+1}}\rangle^{jk}_{*_n}$

whenever (1) x *and* y *are real numbers and* k *and* j *are positive integers*; *or* (2) $x, y \in D_n$, *where* $k, j \in E_n(x)$ (*part a*); $k, kj \in E_n(x)$ (*part b*); $k \in E_n(x) \cap E_n(y)$ (*part c*); $j \in E_n(x)$ *and* $k \in E_n(y)$ (*part d*); *or* $x \in D_{n+1}$, $j > 0, j \in E_n(x)$, *and* $k \in E_{n+1}(x) \cap E_{n+1}(\langle x\rangle^j_{*_n})$ (*part e*).

$*_{n+1}$ and $\langle\ \rangle_{*_n}$ can each also be expressed in terms of the other. (This generalizes an alternate definition $a * b = a^{\ln b} = b^{\ln a}$ that is discussed in Section 5.1 below.)

PROPOSITION 2. *For any* $a, b \in D_n$,

$$a *_{n+1} b = \begin{cases} \langle a\rangle^{\ln^{(n)} b}_{*_n} = \langle b\rangle^{\ln^{(n)} a}_{*_n} & \text{if } n \geq 0 \\ \langle a\rangle^{\exp^{(-n)} b}_{*_n} = \langle b\rangle^{\exp^{(-n)} a}_{*_n} & \text{if } n \leq 0. \end{cases}$$

COROLLARY. *If* $x \in D_n$ *and* $k \in D_{-n}$, *then*

$$\langle x\rangle^k_{*_n} = \begin{cases} x *_{n+1} \exp^{(n)} k & \text{if } n \geq 0 \\ x *_{n+1} \ln^{(-n)} k & \text{if } n \leq 0. \end{cases}$$

Suppose now that $\circ: D_n \times D_n \to D_n$ is continuous and distributes over $*_n$.

LEMMA. $\langle x\rangle^u_{*_n} \circ \langle y\rangle^v_{*_n} = \langle x \circ y\rangle^{u\cdot v}_{*_n}$ *for any* $x, y \in D_n$, $u \in E_n(x)$, $v \in E_n(y)$, *and* $uv \in E_n(x \circ y)$.

Proof. This is proved stepwise, first when u and v are positive integers, then positive rational, then positive real, and finally (when $n \geq 0$) arbitrary real numbers.

After these preliminaries one achieves the main conclusion.

THEOREM 1. *In* $D_n \times D_n$, $*_{n+1}$ *is determined up to a constant by its continuity and its distributivity over* $*_n$: *if* $\circ: D_n \times D_n \to D_n$ *is also continuous and distributes over* $*_n$, *then there is a number* γ *so that* $x \circ y = \langle x *_{n+1} y\rangle^\gamma_{*_n}$ *for all* $x, y \in D_n$.

Proof. When $n \geq 0$, let $u = e_i/\ln^{(n)} x$ and $v = e_j/\ln^{(n)} y$, where i and j are arbitrary positive integers. The lemma then yields $x \circ y = \langle x *_{n+1} y\rangle^\gamma_{*_n}$, where $\gamma = [\ln^{(n)}(e_{i+n} \circ e_{j+n})]/(e_ie_j)$. Similarly, when $n < 0$, set $u = e_i/\exp^{(-n)} x$ and $v = e_j/\exp^{(-n)} y$, satisfying the bounds on E_n by choosing i and $j \geq -n$ so large that $e_ie_j > e_{-n-1} \cdot \exp^{(-n)} x \cdot \exp^{(-n)} y/\exp^{(-n)}(x \circ y)$. The lemma then yields $x \circ y = \langle x *_{n+1} y\rangle^{\gamma}_{*_n}$, where $\gamma = [\exp^{(-n)}(e_{i+n} \circ e_{j+n})]/(e_i e_j)$.

In particular, distributivity and continuity imply both commutativity and associativity.

COROLLARY. *If* $\circ: D_n \times D_n \to D_n$ *is continuous and distributes over* $*_n$, *then it is commutative and associative as well.*

Proof. By the theorem, $x \circ y = y \circ x = \langle x *_{n+1} y\rangle^{\gamma}_{*_n}$, and $(x \circ y) \circ z = x \circ (y \circ z) = \exp^{(n)}[\gamma^2 \cdot \ln^{(n)}x \cdot \ln^{(n)}y \cdot \ln^{(n)}z]$ when $n \geq 0$, or $\ln^{(-n)}[\gamma^2 \cdot \exp^{(-n)}x \cdot \exp^{(-n)}y \cdot \exp^{(-n)}z]$ when $n \leq 0$.

5. REMARKS

This section presents some alternate forms, formulations, and applications of the $*_n$'s.

I. *Interpretations of* $*$ *as Iterated Multiplication.* Multiplication and addition of real numbers are related in three different ways: (1) multiplication distributes over addition, (2) the logarithm of a product is a sum, and (3) multiplication is iterated addition. The new operation $*$ is an appropriate third term in the sequence $+$, $\cdot$, ..., because it has the corresponding relationships to multiplication: (1) $*$ distributes over multiplication (Theorem 3.1),and (2) by definition, $\ln(a * b)$ is $(\ln a) \cdot (\ln b)$. The relationship corresponding to (3) is now shown. An equivalent expression for $*$ is

$$a * b = a^{\ln b} = b^{\ln a} \qquad (a, b > 0). \tag{1}$$

Now when b is a positive integer,

$$a \cdot b = \underbrace{a + a + \cdots + a}_{b \text{ times}} = S_{m(b)}\, a,$$

where S_n is the summation operator adding a to itself n times, and m is a transformation which converts the number b to the instruction to S_- of how many terms are to be summed. $m(b)$ here happens to equal b; the crucial constraint, however, is that m yields a *commutative* operation, so that when a and b are both positive integers,

$$\underbrace{a + a + \cdots + a}_{b \text{ times}} = \underbrace{b + b + \cdots + b}_{a \text{ times}}$$

or

$$S_{m(b)}\, a = S_{m(a)}\, b.$$

For multiplication one considers similarly a product operator P_n (the ordinary power function a^b),

$$\underbrace{a \cdot a \cdot \cdots \cdot a}_{b \text{ times}} = P_b\, a,$$

and seeks a transformation p so that

$$a * b = P_{p(b)}\, a,$$

where

$$P_{p(b)}\, a = P_{p(a)}\, b.$$

By Eq. (1), the logarithm function is precisely the desired transformation.

Another Interpretation. n-fold iterated addition can be conveniently written as multiplication by one fixed term n:

$$S_n(b) = \underbrace{b + b + \cdots + b}_{n \text{ times}} = n \cdot b.$$

Since $e^n * b = b^n$, iterated multiplication can similarly be written in terms of $*$ and one fixed term e^n:

$$P_n(b) = \underbrace{b \cdot b \cdot \cdots \cdot b}_{n \text{ times}} = e^n * b.$$

Equation (1) and the interpretation of $*$ as iterated multiplication are generalized in Propositions 4.1 and 4.2.

II. *$*$ Over the Complex Numbers.* Formula (3.3), which defines $*$ over the reals, can also serve over the complex numbers. Note, however, that there are strong constraints on nontrivial continuous complex operations distributing over multiplication. If $z^n = 1$, then for any y, $z * y = z^n * y^{1/n} = 1 * y^{1/n} = 1$. By continuity, $z * y = 1$ for any z on the unit circle. Consequently, $(re^{i\theta}) * y = (r * y) \cdot (e^{i\theta} * y) = r * y$. Thus $*$ on complex numbers must depend, as in (3.3), on their absolute values, not their arguments.

III. *Third-Quadrant-Negative Operations.* The definition $a *_n^- b = \sigma(a, b) \cdot (a *_n b)$ yields another family of commutative, associative binary operations $*_n^-$ ($n \geq 1$) on the real numbers, with $*_{n+1}^-$ distributing over $*_n^-$ for each $n \geq 1$; like $*^-$, these are all negative in the third quadrant. The two families of operations are, incidentally, "cross-distributive": $*_{n+1}^-$ also distributes over $*_n$, and $*_{n+1}$ over $*_n^-$.

IV. *Generalized Notation.* Iterated sums and products are conveniently represented by the familiar notations Σ and Π. A similar device might be used for iterated $*_n$: Let $\Omega_{i=r}^{s} a_i = a_r *_n a_{r+1} *_n \cdots *_n a_s$. Then $\Sigma_{i=r}^{s} a_i = \Omega_{i=r}^{s} a_i$ and $\Pi_{i=r}^{s} a_i = \Omega_{i=r}^{s} a_i$.

V. *Distributivity Under.* The $*_n$'s distribute *over* addition, multiplication, etc. A complementary problem might ask which operations distribute *under* them. For example, what operations $\circ$ satisfy

$$(a \circ b) \cdot c = (a \cdot c) \circ (b \cdot c) \tag{2}$$

for all real numbers a, b, and c? Ordinary addition is simply the first of an infinite family of operations Δ_n, defined for any odd positive integer n by $x \, \Delta_n \, y = (x^n + y^n)^{1/n}$, which are commutative, associative, and distributed over by multiplication. A general solution follows from the same methods described in the preceding section [1]: If F is a continuous group operation on an interval of real numbers, so that $F(x, y) = h^{-1}[h(x) + h(y)]$, then it distributes over the binary operation G if and only if $G(x, y) = h^{-1}[C[h(x) - h(y)] + h(y)]$ for some continuous function C. Commutativity of G is equivalent to requiring that $C(-t) = C(t) - t$, and associativity of G to $C(C(s) + t) = C(s + t - C(t)) + C(t)$, for all s and t.

The special case of ordinary multiplication yields an interesting relation to bifurcation theory. If $\circ$ satisfies (2), then $a \circ b = a \cdot [1 \circ (b/a)]$ (for $a \neq 0$), and thus $\circ$ is fully determined on nonzero arguments by specifying $f(z) = 1 \circ z$ for each real number z. Associativity of $\circ$, $(a \circ b) \circ c = a \circ (b \circ c)$, is equivalent to

$$f(x) \cdot f(y/f(x)) = f(x \cdot f(y/x)) \qquad (x, f(x) \neq 0), \tag{3}$$

where $x = b/a$ and $y = c/a$. Taking $x = 1$, corresponding to the special case of associativity termed the "left alternative" law [5, p. 370], then yields

$$k \cdot f(t) = f(f(k \cdot t)), \tag{4}$$

where $k = f(1) \neq 0$ and $t = y/k$. It is easy to show that $f(t) = 1 + t$, corresponding to ordinary addition, is the only nontrivial polynomial solution of (4). The equation, though, is a form of the Feigenbaum equation arising in bifurcation theory, and does have other continuous solutions [3, 6].

VI. *Practical Applications.* What sort of applications might the new binary operations $*_n$ have outside abstract algebra? With their numerous singularities, $*_n$'s for large positive n might perhaps be useful for representing divergent or singular phenomena, such as occur in astrophysics or in quantum theories of gravity. Less speculatively, it is discussed here briefly how $*_{-1}$ arises in chemical kinetics. A chemical reaction between two molecules occurs when they collide with sufficient energy to form an activated "transition state" which then decays spontaneously to product species. The rate of the reaction is controlled by the potential energy barrier in this process, termed the activation energy, as described by the equation

$$\Delta G^{\neq} = -RT\ln(kh\mathbf{k}^{-1}T^{-1}),$$

where $\Delta G^{\neq}$ is the change in the Gibbs free energy from the initial to the transition state, k is the rate constant of the reaction, R is the gas constant, T is the absolute temperature, h is Planck's constant, and $\mathbf{k}$ is Boltzmann's constant [4, pp. 47–49]. In multistep reactions, as often occur in biological systems, however, there may not be such a single rate-limiting step and single transition state. Instead, multiple transition states may contribute in ensemble to form a "virtual transition state" that constitutes the overall potential barrier [8, pp. 106–111; 7, p. 966]. The overall $\Delta G^{\neq}$ in such a situation is then related to the $\Delta G_i^{\neq}$'s of the individual steps by

$$\Delta G^{\neq} = RT\ln[\exp[\Delta G_1^{\neq}/(RT)] + \exp[\Delta G_2^{\neq}/(RT)] + \cdots].$$

This complex formalism, however, can now be precisely and simply expressed by the new operation $*_{-1}$ as

$$g = g_1 *_{-1} g_2 *_{-1} \cdots,$$

where $g = \Delta G^{\neq}/(RT)$ and $g_i = \Delta G_i^{\neq}/(RT)$. In the compact notation suggested above, $g = \Omega\, g_i$.

The salient algebraic properties of $*_{-1}$, its associativity and the fact that addition distributes over it, have useful interpretations in this kinetic context. The associativity says that the $\Delta G^{\neq}$ of the overall virtual transition state is independent of the way individual substeps are grouped.

Distributivity yields

$$g - g_0 = (g_1 - g_0) *_{-1} (g_2 - g_0) *_{-1} \cdots$$

or

$$\Delta G^{\neq} - \Delta G_0^{\neq} = RT \ln[\exp[(\Delta G_1^{\neq} - \Delta G_0^{\neq})/(RT)] + \exp[(\Delta G_2^{\neq} - \Delta G_0^{\neq})/(RT)] + \cdots].$$

It thus expresses a property basic for the kinetic analysis, independence from the choice of a reference ground state, in a mathematically transparent way.

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